HISTORIA MATHEMATICA 2 (1975), 31-40

ON THE SOURCES OF MY BOOK MODERNE ALGEBRA

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Motto:

Es steht alles schon bei Dedekind

-- Emmy Noether

SUMMARIES

In December 1971, Garrett Birkhoff asked me to give my view on the main sources for my book. I wrote him a seven-page letter with two supplements. He intended to publish an edited version of my letter, with some commentary of his own, but in the course of our correspondence it turned out that both versions were unsatisfactory. I shall now present an extended record, explaining more fully how I came to write the book and what was the general situation in algebra at that time.

Im Dezember 1971 bat Garrett Birkhoff mich, "to give your view on the main sources for your book." Ich schrieb ihm einen Brief von 7 Seiten und nachher noch zwei Ergänzungen. Er beabsichtigte, eine abgeänderte Fassung meines Briefes mit einem Kommentar zu publizieren, aber im Verlauf unserer Korrespondenz stellte es sich heraus, dass beide Fassungen nicht gut genug waren. Ich werde nun einen erweiterten Bericht vorlegen, in dem ich ausführlicher erkläre, wie ich dazu kam, das Buch zu schreiben, und wie die Situation in der Algebra zu dieser Zeit war.

Introduction

I studied mathematics and physics at the University of Amsterdam from 1919 to 1924. A very nice course in classical algebra was given by Hendrik de Vries. It included subjects like:

determinants and linear equations,
symmetric functions,
resultants and discriminants,
Sturm's theorem on real roots,
Sylvester's "index of inertia" for real quadratic forms,
solution of cubic and biquadratic equations by radicals.

I supplemented this course by studying Galois theory and other subjects in Heinrich Weber's admirable three-volume textbook on algebra. I also read Felix Klein's *Studien über das Ikosaeder* and thoroughly studied the theory of invariants.

In the beginning of our century, many people felt that the theory of invariants was a mighty tool in algebraic geometry. According to Felix Klein's "Erlanger Programm," every branch of geometry is concerned with those properties of geometrical objects that are invariant under a certain group. However, when I studied the fundamental papers of Max Noether, the "Father of Algebraic Geometry" and the father of Emmy Noether, and the work of the great Italian geometers, notably of Severi, I soon discovered that the real difficulties of algebraic geometry cannot be overcome by calculating invariants and covariants.

Already at Amsterdam I pondered over questions of the following kind, without being able to solve them:

How does one define the "dimension" of an algebraic variety?

What do the Italian geometers mean when they speak of a "generic point" (*punto generico*) of a variety?

How can one define intersection multiplicities?

How can one prove the n-dimensional generalizations of Bézout's Theorem on the number of points of intersection of two plane curves?

Can one justify Schubert's "Principle of Conservation of Number," and Schubert's "Calculus of Enumerative Geometry"?

The last was one of Hilbert's problems presented to the Paris Congress in 1900, but I did not know this when I came to Göttingen in 1924.

Another problem that worried me very much was the generalization to n dimensions of Max Noether's "fundamental theorem on algebraic functions." Noether's Theorem specified the conditions under which a given polynomial $F(x,y)$ can be written as a linear combination of two given polynomials f and ϕ with polynomial coefficients A and $B\phi$: $F = Af + B\phi$. More generally, one can ask under what conditions a polynomial $F(x_1,\ldots,x_n)$ can be written as a linear combination of given polynomials $f,\ldots,f_r$, with polynomial coefficients: $F = A_1f_1 + \ldots + A_rf_r$, or in modern terminology, under what conditions is F contained in the ideal generated by $f_1,\ldots,f_r$. From the papers of Max Noether I knew that this question is of considerable importance in algebraic geometry, and I succeeded in solving it in a few special cases. I did not know then that Lasker and Macaulay had obtained much more general results.

Göttingen

When I came to Göttingen in 1924, a new world opened up before me. I learned from Emmy Noether that the tools by which my questions could be handled had already been developed by Dedekind and Weber, by Hilbert, Lasker and Macaulay, by Steinitz and by

Emmy Noether Herself. She told me that I had to study the fundamental paper of E. Steinitz "Algebraische Theorie der Korper" in *Crelle's Journal für die reine und angewandte Mathematik* 137(1910), and Macaulay's Cambridge Tract *Modular Systems*, also the famous paper of Dedekind and Weber on algebraic functions in *Crelle's Journal* 92(1882), and her own papers on ideal theory and elimination theory.

The mathematical library of Göttingen was unique. Everything one needed was there, and one could take the books from the shelves oneself! In Amsterdam and in most continental universities this was impossible. So I started learning abstract algebra and working at my main problem: the foundation of algebraic geometry.

I shall now discuss the main subjects treated in my book, not in the logical order of the text, but approximately in the order in which I learned the theory. In numbering the chapters and sections I shall follow the first edition.

Theory of Fields

In earlier treatises, number fields and fields of algebraic functions were usually treated in separate chapters, and finite fields in still another chapter. The first to give a unified treatment, starting with an abstract definition of "field," was E. Steinitz in his 1910 paper mentioned above. In my Chapter 5, called "Körpertheorie," I essentially followed Steinitz. The proof of the "theorem of the primitive element" in §34 is due to Galois. It was Emmy Noether who drew my attention to this proof.

Chapter 9 deals with infinite field extensions. The main ideas are due, once more, to Steinitz. His proofs were based on well-ordering and transfinite induction; therefore I prefixed Chapter 8, in which these subjects are treated. Sections 57-58, on well-ordering, were drawn from Zermelo's classical papers, but §59, on transfinite induction, was new and was modelled after the treatment of complete induction in §3.

The story of §3 is curious. The main point in §3 is the justification of "definition by complete induction," i.e. the proof of the theorem (the variable x ranges over all natural numbers):

(A) *Given a set of recursive relations defining* $f(x)$ *in terms of the preceding values* $f(m)$ $(m<x)$, *a function* $f(x)$ *exists satisfying these relations.*

Theorem (A) was first proved by Dedekind, essentially as I proved it in §3. His proof is not based upon Peano's axioms. Dedekind presupposes the notion "initial segment from 1 to n," or (which is equivalent) the relation $m<n$. Since this relation can be defined via $m + u = n$, one may also say that Dedekind's proof presupposes addition. In other words, it presupposes the

following theorem:

(B) *A function* $x + y$ *exists satisfying the relations* $x + 1 = x^{+}$, $x + y^{+} = (x+y)^{+}$.

So (B) implies (A), and conversely, for (B) is a special case of (A).

Before 1925, everyone took it for granted that elementary arithmetic, including (A) and (B), follows from Peano's axioms. But in 1927 there were three people who realized that there is a problem here: Edmund Landau, John von Neumann, and Laslo Kálmár.

Landau was preparing his booklet *Grundlagen der Analysis*. He tried to prove (A) or (B) from Peano's axioms, but failed. He discussed the question with John von Neumann, who often came from Berlin to Göttingen. Von Neumann showed that (A) and hence (B) can be derived from Peano's axioms, but his proof was rather complicated. In the same year 1927, Laslo Kálmar visited Göttingen, and showed Landau an extremely simple proof of (B) by complete induction with respect to x . This proof was included in Landau's booklet. In my *Algebra*, I just referred to this booklet for the proof of (B), and I proved (A) by a method due to Dedekind.

Theory of Groups

I learned group theory mainly from Emmy Noether's course "Gruppentheorie und hyperkomplexe Zahlen" (winter 1924/25) and from oral discussions with Artin and Schreier in Hamburg. I also studied Speiser's *Theorie der Gruppen endlicher Ordnung* and Burnside's *Theory of Groups*. Because these excellent textbooks existed, it was not necessary for me to treat the theory of groups in detail in my book.

In Chapter 2, entitled "Gruppen," I restricted myself to those fundamental notions that are used throughout the whole book.

In Chapter 6, entitled "Fortsetzung der Gruppentheorie," the notion of group with operators is introduced, which is used mainly in Chapter 15 ("Lineare Algebra") and in the subsequent chapters 16 and 17. Starting with Dedekind, many authors considered commutative groups with operators, e.g. modules over a ring, but the general notion of group with operators as defined in my book is due to the Russian mathematician Otto Schmidt, who visited Göttingen in 1925 and who published a very nice paper entitled "Ueber unendliche Gruppen mit endlicher Kette" in 1928 (*Mathematische Zeitschrift* 29, p. 34).

Wording and proof of the two isomorphy theorems of §40 are due to Emmy Noether. The same is true for §42.

The proof of the Jordan-Hölder theorem in §41 is due to Otto Schreier. It was published in 1928 (*Abhandlungen aus dem* *mathematischen Seminar Hamburg* 6, p. 300).

The last two sections in Chapter 6 were added because they are needed in the next chapter on Galois theory. The theorems proved in these two sections are due to Galois himself.

In order to follow the historical order, I must now jump to Volume 2.

Theory of Ideals

When I came to Göttingen one of my main problems was the generalization of Max Noether's "fundamental theorem" $F = Af + B\phi$ to n dimensions. The conditions F has to satisfy are "local conditions" in the neighbourhood of the single points of intersection of the curves $f = 0$ and $\phi = 0$. If P is a point of intersection, the local conditions define a "primary ideal" Q, and the original ideal $M - (f,\phi)$ is the intersection of these "primary ideals." The terminology is modern, but the ideas are those of Max Noether and Bertini.

It seems that Hilbert was the first to realize that an n-dimensional generalization of Noether's theorem would be desirable. Emmanuel Lasker, the chess champion, who took his Ph.D. degree under Hilbert's guidance in 1905, was the first to solve this problem. He proved that, quite generally, every polynomial ideal $(f_1,\ldots,f_r)$ is an intersection of primary ideals.

In her 1921 paper "Idealtheorie in Ringbereichen" (*Mathematische Annalen* 83), Emmy Noether generalized Lasker's theorem to arbitrary commutative rings satisfying an "ascending chain condition" (Teilerkettensatz). Chapter 12 of my book *Allgemeine Idealtheorie der kommutativen Ringe* is based on this paper of Emmy Noether. The proof of Hilbert's Finite Basis Theorem in §80 is due to Artin; he presented it in a seminar lecture at Hamburg in 1926. The ascending chain condition is very weak; it is satisfied in all polynomial domains over any field and in many other cases. If stronger assumptions are made concerning the ring, one can even prove that the primary ideals are powers of prime ideals and that every ideal is a product of prime ideals. In Emmy Noether's paper "Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern" (*Mathematische Annalen* 96 (1926), p. 26-61), five axioms were formulated which ensure that every ideal is a product of prime ideals. Rings satisfying these axioms are now called "Dedekind Rings." In these rings Dedekind's theory of ideals in algebraic number fields and fields of algebraic functions of one variable is valid.

The theory of Dedekind fields was presented in Chapter 14 of my book. Emmy Noether's proofs were simplified, making use of an idea of W. Krull contained in §3 of Krull's paper in *Mathematische Annalen* 99 (1927), p. 51-70. Emmy Noether was a referee for this paper, and she told Artin about it. Artin simplified

Krull's proof and presented it in a seminar in Hamburg, in which I participated. Artin's simplified version of Krull's proof was reproduced in §100 (§137 of the paperback edition).

In 1929 I generalized the Dedekind-Noether-Krull-Artin theory to rings integrally closed in their quotient fields. The idea was to replace the ideals by classes of "quasi-equal" ideals. Every ideal was shown to be quasi-equal to a product of prime ideals. This theory was published in *Mathematische Annalen* 101 (1929). In 1930 I received a letter from Artin in which he gave a simpler proof of my theorem, which was reproduced in §103 of my book (§140 of the paperback edition).

Polynomial Ideals and Algebraic Geometry

As I have said already, one of my main concerns was the rigorous foundation of algebraic geometry. The first paper I wrote on this subject, entitled "Nullstellentheorie der Polynomideale," was published in *Mathematische Annalen* 96 (1926). The main part of this paper was incorporated into Chapter 13 of my book. For a fuller account of the history of my earliest papers on algebraic geometry, I may refer to my Nizza lecture on the foundation of algebraic geometry, which was published in *Archive for History of Exact Sciences* 7, p. 171.

In Chapter 13 I also used Macaulay's tract *Modular Systems* Cambridge 1916). The last section (§96) of Chapter 13 is based on my paper "Der Multiplizitätsbegriff der algebraischen Geometrie," *Mathematische Annalen* 97 (1927), in which Schubert's "Principle of Conservation of the Number of Solutions" was proved under fairly general conditions. On the history of this paper see my Nizza lecture just cited, p. 173.

The contents of Chapter 15 ("Lineare Algebra") were generally known in 1924. For §106 I used (and cited) a book of A. Châtelet: *Leçons sur la théorie des nombres* (1913), to which Emmy Noether drew my attention. Section 107 was influenced by Otto Schreier in Hamburg, who was a specialist in linear algebra and theory of groups. Section 108 was drawn from Emmy Noether's paper in *Mathematische Zeitschrift* 30 (1929), p. 641, and §110 was strongly influenced by the classical papers of Frobenius on elementary divisors.

In Chapter 11 ("Eliminationstheorie") §71 and §72 are classical (due to Euler). Sections 73 and 74 are based on the work of the school of Kronecker; my direct source was the Cambridge Tract of F. Macaulay, *Modular Systems*. Section 75 is drawn from A. Rabinowitsch's "Zum Hilbertschen Nullstellensatz," *Mathematische Annalen* 102 (1929), p. 520. His proof of the Nullstellensatz appeared just in time to be included in my book in 1931.

Section 76 was based on the following three papers:

(1) F. Mertens, "Aur Theorie der Elimination I," *Sitzungs-*

berichte der Akademie der Wissenschaften Wien 108 (1899), p. 1173. If my memory is right, it was Ostrowski who drew my attention to this remarkable paper, in which the existence of a system of resultants for homogeneous equations was proved for the first time.

(2) B.L. van der Waerden, "Ein algebraisches Kriterium für die Lösbarkeit eines homogenen Gleichungssystems," *Proceedings Koninklijke Akademie Amsterdam* 29 (1926), p. 142. In this paper, the existence of a system of resultants was proved anew.

(3) H. Kapferer, "Ueber Resultanten und Resultantensysteme," *Sitzungsberichte der Bayerischen Akademie München* 1929, p. 179. In this paper, a shorter proof of the existence of a system of resultants was given.

In §77-78 I mainly used A. Hurwitz, "Ueber die Trägheitsformen eines algebraischen Moduls," *Annali di Matematica* (3a seria) 20 (1913) with simplifications due to myself, "Neue Begründung der Eliminations- und Resultantentheorie," *Nieuw Archief voor Wiskunde* 15 (1928), p. 301.

Section 79 is taken from my paper "Der Multiplizitätsbegriff der algebraischen Geometrie," *Mathematische Annalen* 97 (1927), p. 756.

Summarizing, one may say that the whole of Chapter 11 was closely connected with Emmy Noether's work on elimination theory (*Matematische Annalen* 90, p. 229) and my own work on the foundation of algebraic geometry.

Algebras and Representations

When I came to Göttingen, I took Emmy Noether's course "Gruppentheorie und hyperkomplexe Zahlen" in 1924/25. One of the main subjects in this course was Maclagan Wedderburn's theory of algebras over arbitrary fields. The same subject was treated, in a much improved form, in her course under the same title in 1927/28, in which also a quite new treatment of representations of groups and algebras was given. I took notes of the latter course, and these notes formed the basis of Emmy Noether's publication in *Mathematische Zeitschrift* 30 (1929), p. 641. The Chapters 16 ("Theorie der hyperkomplexen Grössen") and 17 ("Darstellungstheorie der Gruppen und hyperkomplexen Grössen") are almost entirely due to Emmy Noether. Only §127, on the representations of the symmetric groups S_n, comes from an oral communication by John von Neumann, as stated in a footnote.

Göttingen and Hamburg

My first stay in Göttingen lasted just one year, from 1924 to 1925. At this time, the permanent staff of the Mathematics Institute consisted of Hilbert, Herglotz, Landau, Runge, Courant, Emmy Noether and Felix Bernstein. A magnificent constellation!

Among the "Privatdozenten" I mention those to whom I owe most: Alexander Ostrowski, Helmut Kneser, Paul Bernays, and Otto Neugebauer, the historian of science. Prominent guests came from all over the world: Hermann Weyl, Caratheodory, John von Neumann, Siegel, Hasse, Richard Brauer, Heinz Hopf, Paul Alexandroff, Kuratowski, Skolem, Niels and Harald Bohr, Rolf Nevanlinna, Oswald Veblen, G.D. Birkhoff, Norbert Wiener and many others.

I learned mathematical logic mainly from the *Principia Mathematica* of Russell and Whitehead, and set theory from the papers of Felix Bernstein and Zermelo. From Courant and his young pupils Hans Lewy and Kurt Friedrichs I learned the methods of mathematical physics. In topology my masters were Alexandroff, Kuratowski and Kneser; I also studied with great admiration the papers of Alexander. I learned algebraic number theory mainly from the book of Hecke and the famous "Zahlbericht" of Hilbert. In algebraic geometry the papers of Max Noether and those of the great Italian geometers Severi, Castelnuovo and Enriques were a never-failing source of inspiration.

In 1925 I returned to Holland for a year. In 1926 I obtained my Ph.D. with a dissertation in which a program for the foundation of algebraic geometry was developed. Next I went to Hamburg as a Rockefeller fellow to study with Hecke, Artin and Schreier. Artin gave a course on algebra in the summer of 1926. He had promised to write a book on algebra for the "Yellow Series" of Springer. We decided that I should take lecture notes and that we should write the book together. Courant, the editor of the series, agreed. Artin's lectures were marvellous. I worked out my notes and showed Artin one chapter after another. He was perfectly satisfied and said, "Why don't you write the whole book?"

The main subjects in Artin's lectures were fields and Galois theory. In the theory of fields Artin mainly followed Steinitz, and I just worked out my notes. Just so in Galois theory: the presentation given in my book is Artin's.

Of course, Artin had to explain, right at the beginning of his course, fundamental notions such as group, normal divisor, factor group, ring, ideal, field, and polynomial, and to prove theorems such as the Homomorphiesatz and the unique factorization theorems for integers and polynomials. These things were generally known. In most cases I just reproduced Artin's proofs from my notes.

I met Artin and Schreier nearly every day for two or three semesters. I had the great pleasure of seeing how they discovered the theory of "real fields," and how Artin proved his famous theorem on the representation of definite functions as sums of squares. I included all this in my book (Chapter 10). My sources were, of course, the two papers of Artin and Schreier in *Abhandlungen aus dem mathematischen Seminar Hamburg* 5 (1926), p. 83 and 100.

The Introductory Chapters of Volume 1

Chapter 1, "Zahlen und Mengen," was written as an introductory chapter at a time when the rest of Volume 1 was nearly finished. The contents of §1-2 and 4-5 were generally known at that time. The history of §3 was given earlier in this paper.

In *Chapters 2 and 3* I mainly followed the courses of Artin and Noether.

Chapter 4, on polynomials, contains classical material, but §18, on differentiation, is by myself. In §22 ("Irreduzibilitätskriterien") I used the sources indicated in the footnote on page 79 in the first edition: Schönemann, Netto, Dumas, Ore, etc.

Section 23 ("Durchführung der Faktorzerlegung in endlich vielen Schritten") is based on the ideas of Kronecker.

Chapters 5-10 of Volume 1

In the first part of this paper I have already given some explanations concerning Chapters 6 ("Fortsetzung der Gruppentheorie") and 8 ("Ordnung und Wohlordnung von Mengen"). The remaining chapters 5, 7, and 9-10 of Volume 1 all deal with the theory of fields.

In *Chapter 5* ("Körpertheorie") I mainly followed Artin and Steinitz. For §35 I used Noether's course on hypercomplex systems.

Section 37 is new. Grete Herrmann, a pupil of Emmy Noether, had treated the same problem in her dissertation. In her treatment there was a lacuna, which I pointed out in a paper "Eine Bemerkung über die Unzerlegbarkeit von Polynomen," *Mathematische Annalen* 102 (1930), p. 738. Grete Herrmann was not trained in intuitionistic mathematics, whereas I was, because I had studied under the guidance of L.E.J. Brouwer at Amsterdam. Therefore, I noticed her error at once and, using Brouwer's methods, I constructed a counter-example to one of her statements.

Chapter 7 on Galois theory was based on Artin's course of lectures (see my paper "Die Galois-Theorie von Heinrich Weber bis Emil Artin," *Archive for the History of Exact Sciences* 9 (1972), p. 240. Only §54 and §56 (§65 and §66 in the paperback edition) were added by myself.

Chapter 9 was entirely taken from the classical paper of E. Steinitz, "Algebraische Theorie der Körper," *Journal für die reine und angewandte Mathematik* 137 (1910).

In *Chapter 10*, two subjects were combined, which in later editions were treated in separate chapters, namely:

(a) the Artin-Schreier theory of real fields and the representation of positive rational functions as sums of squares (§81-83 of the paperback edition, §68-70 of the first edition).

(b) fields with valuations and p-adic fields (§65 of the first edition, Chapter 18 in the paperback).

In treating subject (a), I closely followed the papers of

Artin and Schreier quoted before. I also used a paper by R. Baer, "Ueber nichtarchimedisch geordnete Körper," *Sitzungsberichte der Heidelberger Akademie, Abhandlung 8* (1927). As an introduction, I added §64, "Definition der reellen Zahlen." This §64 was inspired by Cantor's construction of real numbers as "Fundamentalfolgen," but written in such a way that the generalization to fields with valuations becomes obvious. This generalization was presented in §65.

For part (b), that is for §65, the main sources were the papers of Hensel, Kürschak and Ostrowski on p-adic fields and fields with valuations quoted on p. 220 (Vol. 1 of the first edition). Note that Ostrowski was at Göttingen when I wrote Volume 1, and Hasse, Hensel's best and a great propagandist of p-adic methods, often came to Gottingen. More papers on valuations are quoted on p. 206 of the paperback edition of Volume 2, and in my paper "Algebra seit Galois," *Jahresbericht der deutschen Mathematiker-Vereinigung* 68 (1966), p. 155.

In later editions, when the importance of valuations became more and more obvious, a separate chapter was devoted to fields with valuations (Chapter 18 in the paperback edition). The main source was Ostrowski's most important paper on valuations in *Mathematische Zeitschrift* 39 (1934), p. 296-404.

Still later, a chapter on topological algebra was added, containing the theory of topological groups, rings, and fields. The first to develop these theories in a systematic way was my friend D. van Dantzig in Amsterdam. In the fourth and later editions I mainly based myself on van Dantzig's fundamental paper "Zur topologischen Algebra I: Komplettierungstheorie," *Mathematische Annalen* 107 (1933), p. 587. I also used the papers of Kaplanski, Kowalski, and Pontryagin cited on page 292 of the paperback edition of Volume 2.

Another later addition to Volume 2 (first in the fourth edition, 1959) was the chapter "Algebraische Funktionen einer Variablen" (Chapter 19 in the paperback edition). This chapter culminates in a proof of the Riemann-Roch theorem based on the ideas of Dedekind and Weber, Emmy Noether, F.K. Schmidt, Severi, and André Weil. For the history of this proof, see the introduction of Chapter 19 in the paperback edition.

Also in the fourth edition, the chapter on algebras (Chapter 13 in the paperback edition) was considerably enlarged, and the proofs were simplified by combining the original methods of Emmy Noether with those of Jacobson. In §93 of the paperback edition subsections on the algebras of Grassmann and Clifford were added. The sources of these subsections were listed at the end of §93 (Vol. 2, p. 42 of the paperback edition).